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Stress indices of graphs

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Abstract

The stress of a vertex is the number of shortest paths passing through it. The stess indices of cycle C_n and of wheel W_{n+1} are existing in terms of odd n and even n separately. In this paper we obtain stress indices of cycle and wheel in terms of single expressions of n. Further we obtain the stress indices of trees considering all its vertices.

Keywords: Stress of a vertex, Stress indices of graphs, Stress-sum index.

2020 MSC: 05C09, 05C12.

1. Introduction

Centrality measure plays a very important role in the study of network analysis [1, 2]. In 1953, Shimbel [3] defined the measure of stress centrality based on the shortest paths. Stress centrality measure has lot of applications in social and biological networks.

Let G be a simple, connected graph with n vertices and m edges. Let $V(G)$ be the vertex set and $E(G)$ be the edge set of of G. The line joining the vertices u and v is denoted by uv.

The stress of a veretx u is defined as the the number of shortest paths passing through it and is denoted by $st(u)$ [3]. The stress of a graph G is defined as the sum of the stresses of all its vertices and is denoted by $st(G)$. Rajendra et al. [4] defined the first stress index $S_1(G)$ and second stress index $S_2(G)$ as

$$S_1(G) = \sum_{v \in V(G)} st(v)^2$$

and

$$S_2(G) = \sum_{uv \in E(G)} st(u)st(v).$$

Further the stress sum index $SS(G)$ of a graph is defined as [5]

$$SS(G) = \sum_{uv \in E(G)} [st(u) + st(v)].$$

In [4, 5], the stress indices of the complete bipartite graph, cycle, path, tree, windmill graph and of wheel have been computed. The stress indices of cycle C_n and wheel W_{n+1} are computed for odd n and even n separately. In this paper we obtain stress indices of cycle, wheel and fan in terms of single expressions of n. Also the stress indices of trees are obtained in [4, 5] by considering non-pendent vertices. We obtain here the stress indices of trees considering all its vertices.

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2. Stress indices of graphs

Theorem 2.1. For a cycle C_n on $n \geqslant 3$ vertices,

$$\begin{aligned} \text{(i)} \quad S_1(C_n) &= \frac{1}{4}nd^2(d-1)^2 \\ \text{(ii)} \quad S_2(C_n) &= \frac{1}{4}nd^2(d-1)^2 \\ \text{(iii)} \quad SS(C_n) &= nd(d-1), \end{aligned}$$

where $d = \lfloor \frac{n}{2} \rfloor$.

Proof. For any vertex u of a cycle C_n, we have $st(u) = \frac{d(d-1)}{2}$, where $d = \lfloor \frac{n}{2} \rfloor$. Therefore

$$\begin{aligned} \text{(i)} \quad S_1(C_n) &= \sum_{u \in V(C_n)} st(u)^2 \\ &= \sum_{u \in V(C_n)} \frac{1}{4}d^2(d-1)^2 \\ &= \frac{1}{4}nd^2(d-1)^2. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad S_2(C_n) &= \sum_{uv \in E(C_n)} st(u)st(v) \\ &= \sum_{uv \in E(C_n)} \left(\frac{d(d-1)}{2}\right)\left(\frac{d(d-1)}{2}\right) \\ &= \frac{1}{4}nd^2(d-1)^2. \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad SS(C_n) &= \sum_{uv \in E(C_n)} [st(u) + st(v)] \\ &= \sum_{uv \in E(C_n)} \left[\frac{d(d-1)}{2} + \frac{d(d-1)}{2}\right] \\ &= nd(d-1). \end{aligned}$$

□

A wheel W_{n+1}, $n \geqslant 3$ is obtained from the cycle C_n by adding a new vertex and making it adjacent to all vertices of C_n.

Theorem 2.2. For a wheel W_{n+1}, $n \geqslant 4$,

$$\begin{aligned} \text{(i)} \quad S_1(W_{n+1}) &= n + \frac{1}{4}n^2(n-3)^2 \\ \text{(ii)} \quad S_2(W_{n+1}) &= \frac{n}{2}(n^2 - 3n + 2) \\ \text{(iii)} \quad SS(W_{n+1}) &= \frac{n}{2}(n^2 - 3n + 6). \end{aligned}$$

Proof. Let w be the central vertex of W_{n+1} and $v_1, v_2, \ldots, v_n$ be the vertices on the cycle C_n of the wheel. We have $st(w) = \frac{n(n-3)}{2}$ and $st(v_i) = 1$ for $i = 1, 2, \ldots, n$. Therefore

$$\begin{aligned}
\text{(i)}\quad S_1(W_{n+1}) &= \sum_{u \in V(W_{n+1})} st(u)^2 \\
&= st(w)^2 + \sum_{i=1}^{n} st(v_i)^2 \\
&= \frac{n^2(n-3)^2}{4} + n.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad S_2(W_{n+1}) &= \sum_{uv \in E(W_{n+1})} st(u)st(v) \\
&= \sum_{i=1}^{n-1} st(v_i)st(v_{i+1}) + st(v_n)st(v_1) + \sum_{i=1}^{n} st(w)st(v_i) \\
&= 1(n-1) + 1 + \frac{n^2(n-3)}{2} \\
&= \frac{n}{2}(n^2 - 3n + 2).
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad SS(W_{n+1}) &= \sum_{uv \in E(W_{n+1})} [st(u) + st(v)] \\
&= \sum_{i=1}^{n-1} [st(v_i) + st(v_{i+1})] + [st(v_n) + st(v_1)] + \sum_{i=1}^{n} [st(w) + st(v_i)] \\
&= 2(n-1) + 2 + n\left(1 + \frac{n(n-3)}{2}\right) \\
&= \frac{n}{2}(n^2 - 3n + 6).
\end{aligned}$$

□

A fan F_{n+1} is obtained from the path P_n by adding a new vertex and making it adjacent to all vertices of P_n.

Theorem 2.3. For a fan F_{n+1}, $n \geqslant 3$,

$$\begin{aligned}
\text{(i)}\quad S_1(F_{n+1}) &= (n-2)\left[1 + \frac{1}{4}(n-1)^2(n-2)\right] \\
\text{(ii)}\quad S_2(F_{n+1}) &= (n-3) + \frac{1}{2}(n-1)(n-2)^2 \\
\text{(iii)}\quad SS(F_{n+1}) &= 3n - 6 + \frac{1}{2}n(n-1)(n-2).
\end{aligned}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the path P_n and w be the vertex of degree n in fan F_{n+1}. We have $st(w) = \frac{(n-1)(n-2)}{2}$, $st(v_1) = st(v_n) = 0$ and $st(v_i) = 1$ for $i = 2, 3, \ldots, n-1$. Therefore

$$\begin{aligned}
\text{(i)}\quad S_1(F_{n+1}) &= \sum_{u\in V(F_{n+1})} st(u)^2 \\
&= st(w)^2 + \sum_{i=1}^{n} st(v_i)^2 \\
&= \frac{1}{4}(n-1)^2(n-2)^2 + (n-2) \\
&= (n-2)\left[1+\frac{1}{4}(n-1)^2(n-2)\right].
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad S_2(F_{n+1}) &= \sum_{uv\in E(F_{n+1})} st(u)st(v) \\
&= \sum_{i=1}^{n-1} st(v_i)st(v_{i+1}) + \sum_{i=1}^{n} st(w)st(v_i) \\
&= (n-3) + (n-2)\frac{(n-1)(n-2)}{2} \\
&= (n-3) + \frac{1}{2}(n-1)(n-2)^2.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad SS(F_{n+1}) &= \sum_{uv\in E(F_{n+1})} [st(u)+st(v)] \\
&= \sum_{i=1}^{n-1} [st(v_i)+st(v_{i+1})] + \sum_{i=1}^{n} [st(w)+st(v_i)] \\
&= 1+2(n-3)+1+2\left[\frac{1}{2}(n-1)(n-2)+0\right] + (n-2)\left[\frac{1}{2}(n-1)(n-2)+1\right] \\
&= 3n-6+\frac{1}{2}n(n-1)(n-2).
\end{aligned}$$

□

Theorem 2.4. Let T be a tree with n vertices. Let u and v be the vertices of T. Let $T_1, T_2, \ldots, T_{k_1}$ be the components of $T-u$, where $|V(T_i)| = n_i$, $i = 1,2,\ldots,k_1$ and let $T_1', T_2', \ldots, T_{k_2}'$ be the components of $T-v$, where $|V(T_i')| = n_i'$, $i = 1,2,\ldots,k_2$. Then

$$\begin{aligned}
\text{(i)}\quad S_1(T) &= \frac{n(n-1)^4}{4} - \frac{(n-1)^2}{2}\sum_{u\in V(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right) + \frac{1}{4}\sum_{u\in V(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right)^2 \\
\text{(ii)}\quad S_2(T) &= \frac{(n-1)^5}{4} - \frac{(n-1)^2}{4}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2 + \sum_{i=1}^{k_2} {n_i'}^2\right) + \frac{1}{4}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right)\left(\sum_{i=1}^{k_2} {n_i'}^2\right) \\
\text{(iii)}\quad SS(T) &= (n-1)^3 - \frac{1}{2}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2 + \sum_{i=1}^{k_2} {n_i'}^2\right).
\end{aligned}$$

Proof. For any vertex u of a tree T [6],

$$st(u) = \frac{1}{2}\left[(n-1)^2 - \sum_{i=1}^{k_1} {n_i}^2\right].$$

Therefore

$$\begin{aligned}
\text{(i)}\quad S_1(T) &= \sum_{u\in V(T)} st(u)^2 \\
&= \sum_{u\in V(T)} \frac{1}{4}\left[(n-1)^2 - \sum_{i=1}^{k_1} {n_i}^2\right]^2 \\
&= \frac{n(n-1)^4}{4} - \frac{(n-1)^2}{2}\sum_{u\in V(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right) + \frac{1}{4}\sum_{u\in V(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right)^2.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad S_2(T) &= \sum_{uv\in E(T)} st(u)st(v) \\
&= \sum_{uv\in E(T)} \frac{1}{4}\left((n-1)^2 - \sum_{i=1}^{k_1} {n_i}^2\right)\left((n-1)^2 - \sum_{i=1}^{k_2} {n_i'}^2\right) \\
&= \frac{(n-1)^5}{4} - \frac{(n-1)^2}{4}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2 + \sum_{i=1}^{k_2} {n_i'}^2\right) + \frac{1}{4}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2\right)\left(\sum_{i=1}^{k_2} {n_i'}^2\right).
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad SS(T) &= \sum_{uv\in E(T)} [st(u) + st(v)] \\
&= \sum_{uv\in E(T)} \left[\frac{1}{2}\left((n-1)^2 - \sum_{i=1}^{k_1} {n_i}^2\right) + \frac{1}{2}\left((n-1)^2 - \sum_{i=1}^{k_2} {n_i'}^2\right)\right] \\
&= (n-1)^3 - \frac{1}{2}\sum_{uv\in E(T)}\left(\sum_{i=1}^{k_1} {n_i}^2 + \sum_{i=1}^{k_2} {n_i'}^2\right).
\end{aligned}$$

□

Theorem 2.5. Let G be a connected regular graph of degree r. Then $SS(G) = rst(G)$.

Proof. We have

$$\begin{aligned}
SS(G) &= \sum_{uv\in E(G)} [st(u) + st(v)] \\
&= \sum_{v\in V(G)} deg(v)st(v) \\
&= r\sum_{v\in V(G)} st(v) \\
&= rst(G).
\end{aligned}$$

□

3. Conclusion

In [4, 5], the stress indices of cycle C_n and wheel W_{n+1} are computed in terms of odd n and even n separately. We have obtained stress indices of cycle, wheel and fan in terms of single expressions of n. The stress indices of trees are obtained in [4, 5] by considering non-pendent vertices. We have obtained the stress indices of trees considering all its vertices.

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