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Sombor and Second Zagreb Indices of Total Generalized SIERPIŃSKI Gasket Graph

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Abstract

Sierpiński fractals and Sierpiński gasket graphs have many applications in diverse areas like dynamical systems, chemistry and problem. In this paper, we study and determine the Sombor index and second Zagreb index for the total generalized Sierpiński gasket of paths, Cycles, and Complete Graphs.

Keywords: Total Generalized Sierpiński Gasket graph, Sombor index and Zagreb index.

2020 MSC: 05C05, 05C07.

1. Introduction

Throughout this paper, we consider simple connected graphs. Let $G = (V, E)$ be a graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. In mathematical chemistry and chemical graph theory, a topological index is a numerical parameter (a real number) that is measured based on the molecular graph of a chemical constitution [1]. One of the important topological indices introduced about forty years ago by Ivan Gutman and Trinajstic [2] is the second Zagreb index $M_2(G)$ which is defined as:

$$M_2(G) = \sum_{uv \in E(G)} \deg(u)\deg(v).$$

Also Gutman in [3] defined a new vertex-degree-based graph invariant, named "Sombor index" for a graph G, denoted by $SO(G)$, as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2}$$

Mathematical properties and applications of the SO index were established in [3]. Decomposition into special substructures that inherit remarkable features is an important method used for the investigation of some mathematical structures, specifically when the regarded structures have self-similarity features. In these cases, we usually only need to study the substructures and the way that they are related to each other. Klavzar et al. for the first time, introduced the Sierpiński graph $S(K_n, t)$, see [4] and [5]. One of the most important families of these self-similar graphs is the family of Sierpiński gasket graphs, see [6].

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Definition 1.1. Let G be a graph of order $n \geqslant 2$, with the vertex set $V = \{1, 2, \dots n\}$ and t be a positive integer. If l is adjacent to j in G, then by contracting the new edge between two copies l and j (the linking edge) in the generalized Sierpiński graph, the total generalized Sierpiński gasket graph is obtained. In other words, when j is adjacent to l in G, the vertex $\mathbf{u} = v_1 v_2 \dots v_r j l \dots l$ is adjacent to $\mathbf{v} = v_1 v_2 \dots v_r l j \dots j$, in $S(G, t)$, $0 \leqslant r \leqslant t-2$, the edge $\mathbf{uv}$ will be contracted in $SG[G, t]$, and this new vertex will be denoted by $v_1 v_2 \dots v_r \{j, l\}_{t-r}$ or shortly by $v_{(r)}\{j, l\}_{t-r}$, see Figure 1.

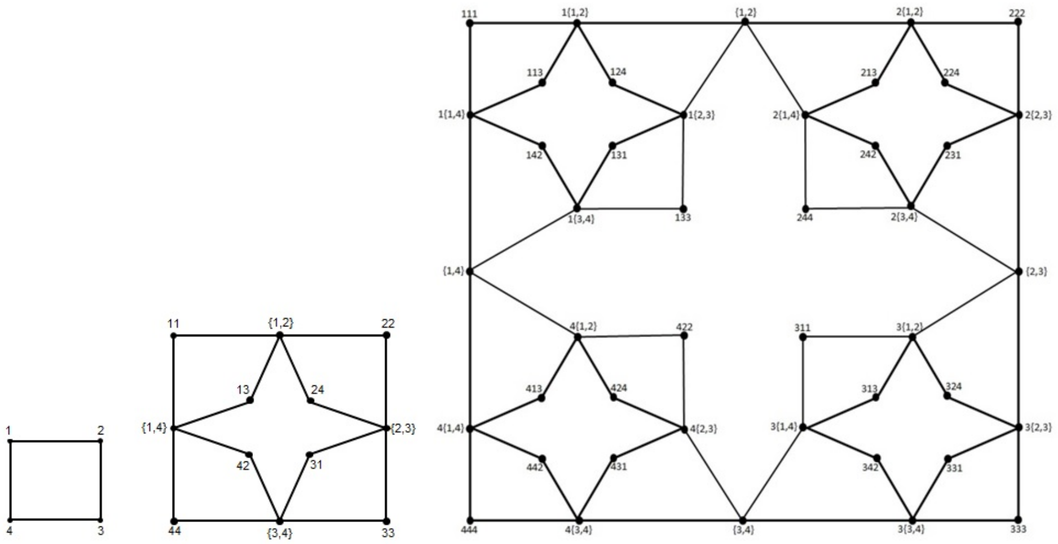

Figure 1: Total generalized Sierpiński gasket graphs C_4, $SG[C_4, 2]$ and $SG[C_4, 3]$.

Remark 1.2. Similar to the structure of the generalized Sierpiński graph $S(G, t)$, $SG[G, t]$ is constructed inductively by inserting a copy of $SG[G, t-1]$ instead of each vertex of G ($SG_i[G, t]$ for $i \in V(G)$) and then by contracting the new $|E(G)|$ linking edges (of $S(G, t)$). More precisely, when i is adjacent to j in the graph G, then the linking edge between $ijj \dots j$ and $jii \dots i$ is contracted and the new vertex is shown by $\{i, j\}_t$ in $SG[G, t]$. Note that the vertex $\{i, j\}_t$ is the unique common shared vertex between two copies $SG_i[G, t]$ and $SG_j[G, t]$.

Remark 1.3. The Total Generalized Sierpiński Gasket graph is the graph with vertex set V^t and the non-contracted vertices $\mathbf{u} = u_1 u_2 \dots u_t$ and $\mathbf{v} = v_1 v_2 \dots v_t$ are adjacent in $SG[G, t]$ if and only if

(i) $u_i = v_i \quad \text{for} \quad i \neq t,$

(ii) $u_t \neq v_t \quad \text{and} \quad u_t v_t \in E(G).$

For contracted vertices, it is enough to consider the expanded forms of these vertices. When $\mathbf{u} = u_1 u_2 \dots u_t$ and $\mathbf{v} = v_1 v_2 \dots v_t$, then the edge $\mathbf{uv}$ is contracted in $SG[G, t]$, if and only if there exists $i \in \{1, \dots, t\}$ such that:

(i) $u_j = v_j$ if $j < i,$

(ii) $u_i \neq v_i$ and $u_i v_i \in E(G),\ i \neq t,$

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i.$

2. Main results

In this section, we determine some of the topological indices of $SG[G, t]$ for special graphs G in step t.

Theorem 2.1. *The second Zagreb index of total generalized Sierpiński gasket graph of* K_n *in step* t *is given by*

$$M_2(SG[K_n, t]) = 2n(-n^3 + 3n^2 - 3n + 1) + 4mn^{t-1}(n^2 - 2n + 1).$$

Proof. By considering Remark 1.3, in $SG[K_n, t]$ we have n extreme vertices in form $ii \ldots i$, $1 \leqslant i \leqslant n$, and other vertices are in contracted form. Since each extreme vertex has degree $n-1$ and its neighbours have degree $2(n-1)$, we have

$$\begin{aligned} M_2(SG[k_n, t]) &= \sum_{uv \in SG[k_n,t]} \deg(u)\deg(v) \\ &= (n^2-n)((n-1)(2n-2)) + (mn^{t-1}-n^2+n)(2n-2)^2 \\ &= 4mn^{t-1}(n^2-2n+1) + 2n(-n^3+3n^2-3n+1). \end{aligned}$$

□

Theorem 2.2. *For the Sombor index of* $SG[K_n, t]$ *we have*

$$SO(SG[K_n, t]) = (n^2-n)(\sqrt{5n^2-10n+5}) + 2\sqrt{2}n(mn^{t-1} - mn^{t-2} - n^2 + 2n - 1).$$

Proof. Since in $SG[k_n, t]$ there are $n(n-1)$ edges whose endpoints have degree $(n-1)$ and $2(n-1)$, and the other edges have two endpoints of degree $2(n-1)$, we get

$$\begin{aligned} SO(SG[k_n, t]) &= \sum_{uv \in SG[k_n,t]} \sqrt{\deg(u)^2 + \deg(v)^2} \\ &= (n^2-n)\sqrt{(n-1)^2+(2n-2)^2} + (mn^{t-1}-n^2+n)(\sqrt{2(2n-2)^2}) \\ &= (n^2-n)(\sqrt{5n^2-10n+5}) + 2\sqrt{2}n(mn^{t-1}-mn^{t-2}-n^2+2n-1). \end{aligned}$$

□

Theorem 2.3. *If* $n \geqslant 4$ *and* $t \geqslant 2$, *then*

$$M_2(SG[C_n, t]) = \frac{4n(n^t + 3n^{t-1} + 4n^{t-2} - 8)}{n-1}.$$

Proof. we have three types of edges, there are $n^{t-1}(n-4)$ edges whose two endpoints have degree 2, and $\frac{4n(1-n^{t-2})}{1-n}$ edges whose endpoints have degree 4, and $\frac{4n+n^{t-1}(4n-8)}{n-1}$ edges whose endpoints have degree 2 and 4. Thus,

$$\begin{aligned} M_2(SG[C_n, t]) &= \sum_{uv \in SG[C_n,t]} \deg(u)\deg(v) \\ &= n^{t-1}(n-4)(2\times 2) + \frac{4n(1-n^{t-2})}{1-n}(4\times 4) + \frac{4n+n^{t-1}(4n-8)}{n-1}(2\times 4) \\ &= \frac{4n(n^t+3n^{t-1}+4n^{t-2}-8)}{n-1}. \end{aligned}$$

□

Theorem 2.4. *For each* $n \geqslant 4$ *and* $t \geqslant 2$, *the Sombor index of* $SG[C_n, t]$ *is obtained*

$$SO(SG[C_n, t]) = \frac{2\sqrt{2}n(n^t - 5n^{t-1} + 12n^{t-2} - 8) + 8\sqrt{5}n(n^{t-1} - 2n^{t-2} + 1)}{n-1}.$$

Proof. By considering the proof of Theorem 2.3, we have

$$\begin{aligned} SO(SG[C_n,t]) &= \sum_{uv\in SG[C_n,t]} \sqrt{\deg(u)^2+\deg(v)^2} \\ &= n^{t-1}(n-4)\sqrt{2^2+2^2}+\frac{4n(1-n^{t-2})}{1-n}\sqrt{4^2+4^2}+\frac{4n+n^{t-1}(4n-8)}{n-1}\sqrt{2^2+4^2} \\ &= \frac{2\sqrt{2}n(n^t-5n^{t-1}+12n^{t-2}-8)+8\sqrt{5}n(n^{t-1}-2n^{t-2}+1)}{n-1}. \end{aligned}$$

□

Theorem 2.5. *If $n \geqslant 4$, then the second Zagreb index of total generalized Sierpiński gasket graph of the path P_n in step 2 is determined by*

$$M_2(SG[P_n,2]) = 94+4(n-3)(n-4)+4(n-5)+32(m-4).$$

Proof. The second Zagreb index of total generalized sierpinski gasket graph P_n in step 2 is as follows:

$$\begin{aligned} M_2(SG[P_n,2]) &= \sum_{uv\in SG[P_n,2]} \deg(u)\deg(v) \\ &= 2(1\times 3)+2(1\times 4)+(n-3)(n-4)(2\times 2)+(2+2)(3\times 2)+(2n-6)(1\times 2) \\ &+ (2\times 3)(4\times 2)+4(m-4)(2\times 4) \\ &= 94+4(n-3)(n-4)+4(n-5)+32(m-4). \end{aligned}$$

□

Theorem 2.6. *If $n \geqslant 4$, then the Sombor index of $SG[P_n,2]$ is equal to*

$$SO(SG[P_n,2]) = 2\sqrt{10}+2\sqrt{17}+2\sqrt{2}(n-3)(n-4)+4\sqrt{13}+2\sqrt{5}(4m+n-13)$$

Proof. The Sombor index of $SG[P_n,2]$ is as follows:

$$\begin{aligned} SO(SG[P_n,2]) &= \sum_{uv\in SG[P_n,2]} \sqrt{\deg(u)^2+\deg(v)^2} \\ &= 2\sqrt{1^2+3^2}+2\sqrt{1^2+4^2}+(n-3)(n-4)\sqrt{2^2+2^2}+(2+2)\sqrt{3^2+2^2} \\ &+ (2n-6)\sqrt{1^2+2^2}+(2\times 3)\sqrt{2^2+4^2}+4(m-4)\sqrt{2^2+4^2} \\ &= 2\sqrt{10}+2\sqrt{17}+2\sqrt{2}(n-3)(n-4)+4\sqrt{13}+2\sqrt{5}(4m+n-13). \end{aligned}$$

□

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