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Multi-Super-Stability of Ternary Antiderivation in Ternary Banach Algebras

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Abstract

In this study, we introduce the notion of ternary antiderivation on ternary Banach algebras and investigate the multi-super-stability of ternary antiderivation in ternary Banach algebras, associated with functional inequalities.

Keywords: Hyers-Ulam stability, multi-super-stability, fixed point method, ternary antiderivation, ternary Banach algebra, additive functional inequality.

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1. Introduction

First, we recall a fundamental result in fixed point theory.

Assume Banach algebras $\mathscr{X}$ and $\mathscr{X}''$. Suppose $(\mathscr{X}', \Delta)$ is a probability measure space and suppose $(\mathscr{X}, \mathfrak{B}_{\mathscr{X}})$ and $(\mathscr{X}'', \mathfrak{B}_{\mathscr{X}''})$ are Borel measurable spaces. Then a map $\mathscr{J} : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}''$ is a operator if $\{\mathfrak{P} : \ \mathscr{J}(\mathfrak{P}, \alpha) \in \nu\} \in \Delta$ for each α in $\mathscr{X}$ and $\nu \in \mathfrak{B}_{\mathscr{X}''}$. Now, we are going to propose vector valued generalized metric spaces. Assume $\mho = (\mho_1, \ldots, \mho_m)$ and $\Omega = (\Omega_1, \ldots, \Omega_m), m \in \mathbb{N}$. Then we have

$$\mho \preceq \Omega \Longleftrightarrow \mho_\iota \leqslant \Omega_\iota, \quad \iota = 1, \cdots, m;$$

and also

$$\mho \to 0 \Longleftrightarrow \mho_\iota \to 0, \quad \iota = 1, \cdots, m.$$

Definition 1.1 ([1]). Let $\nabla \neq \emptyset$ is a set and $d : \nabla^2 \to [0, +\infty]^m, m \in \mathbb{N}$, is a given mapping. If the following conditions are satisfied, then we say d is a generalized metric on ∇:

▷ for each $(g, g') \in \nabla \times \nabla$, we get

$$d(g, g') = \underbrace{(0, \cdots, 0)}_{m} \Longleftrightarrow \ g = g';$$

▷ for each $(g, g') \in \nabla \times \nabla$, we get

$$d(g', g) = d(g, g') \Longleftrightarrow \ g = g';$$

▷ for each $g, g', g'' \in \nabla$, we get

$$d(g, g'') + d(g'', g') \succeq d(g', g).$$

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Theorem 1.2 ([1]). Assume the following assumptions:

▷ $d : \nabla^2 \to [0, +\infty]^m$, $m \in \mathbb{N}$, and (∇, d) is a complete generalized metric space.

▷ $\mathscr{L} : \nabla \to \nabla$ is a strictly contractive mapping with Lipschitz constant $\mathcal{Z} < 1$.

Then for each $g \in \nabla$, either

$$d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) = \overbrace{(+\infty, \cdots, +\infty)}^{m}$$

for each $n \in \mathbb{N} \cup \{0\}$ or there is a $n_0 \in \mathbb{N}$ such that

▷ $d(\mathscr{L}^n g, \mathscr{L}^{n+1} g) \preceq \overbrace{(+\infty, \cdots, +\infty)}^{m}, \qquad \forall n \geqslant n_0;$

▷ the sequence $\{\mathscr{L}^n g\}$ converges to a fixed point $(g')^*$ of $\mathscr{L}$;

▷ $(g')^*$ is the unique fixed point of $\mathscr{L}$ in the set $\complement = \{g' \in \nabla \mid d(\mathscr{L}^{n_0} g, g') \preceq \overbrace{(+\infty, \cdots, +\infty)}^{m}\}$;

▷ $d(g', (g')^*) \preceq \frac{1}{1-\mathcal{Z}} d(g', \mathscr{L} g')$ for each $g' \in \complement$.

Now, by introducing some special functions, we present the concept of multi-stability.

Consider the following special functions [1].

•The Gauss hypergeometric series :

$$\varphi_1^{\circledS}(X) := {}_2\mathbb{F}_1(\alpha, B; T; X) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (B)_n}{(T)_n} \frac{X^n}{n!},$$

where $\alpha, B, T, X \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, and $|X| < 1$.

• The Clausen hypergeometric series :

$$\varphi_2^{\circledS}(X) := {}_p\mathbb{F}_q((\alpha); (T); X) = {}_p\mathbb{F}_q\left({}^{\alpha_1, \cdots, \alpha_p}_{T_1, \cdots, T_q}; X\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(T_1)_k \cdots (T_q)_k} \frac{X^n}{n!}, \tag{1.1}$$

where $p, n, q \in \mathbb{N} \cup \{0\}$ and $\alpha_n, X, T_n \in \mathbb{C}$.

• The Wright generalized hypergeometric function. Assume $\Xi := -\sum_{k=1}^{q} b_k + \sum_{j=1}^{p} a_j$, $\sigma := -\prod_{k=1}^{q} |b_k|^{-b_k} + \prod_{j=1}^{p} |a_j|^{-a_j}$, and $\chi := -\sum_{j=1}^{p} \kappa_j + \sum_{k=1}^{q} \vartheta_k + \dfrac{p-q}{2}$, where $\kappa_j, \vartheta_k \in \mathbb{C}$, $k, j \in \mathbb{N}$, $p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

Therefore Wright generalized hypergeometric series is given by

$$\varphi_3^{\circledS}(X) = {}_p\mathbb{W}_q\left({}^{(\kappa_p, a_p)_{1,p}}_{(\vartheta_q, b_q)_{1,q}}; X\right) = \sum_{s=0}^{\infty} \frac{\left\{\prod_{j=1}^{p} \Gamma(\kappa_j + a_j s)\right\}}{\left\{\prod_{k=1}^{q} \Gamma(\vartheta_k + b_k s)\right\}} \frac{X^s}{s!}, \tag{1.2}$$

where $j, s, k \in \mathbb{N}, X \in \mathbb{C}$, $\Xi > -1, \kappa_j, \vartheta_k \in \mathbb{C}$, $p, q \in \mathbb{N} \cup \{0\}$, and $b_k, a_j \in \mathbb{R}_+$.

• The Wright function :

$$\varphi_4^{\circledS}(X) := \mathbb{K}(\vartheta, b, X) = {}_0\mathbb{W}_1\left({}^{-}_{(b,\vartheta)}; X\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\vartheta + bk)} \frac{X^k}{k!}, \tag{1.3}$$

where $X, \vartheta \in \mathbb{C}$, and $b \in \mathbb{R}$.

• The Wright generalized Bessel function (Bessel–Maitland function) :

$$\varphi_5^{\circledS}(X) := \mathbb{J}(\kappa, a, X) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\kappa + 1 + ak)} \frac{(-X)^k}{k!} = {}_0\mathbb{W}_1\left({}^{-}_{(\kappa+1,b)}; -X\right),$$

where $\kappa, X \in \mathbb{C}$, and $a \in \mathbb{R}$.

•The shifted Wright generalized hypergeometric series :

$$\varphi_6^{\circledS}(X) = {}_p\mathbb{B}_q\left({(\kappa_p,a_p;\vartheta_p,b_p)_{1,p} \atop (\widehat{\kappa}_p,c_p;\widehat{\vartheta}_p,d_p)_{1,q}};X\right) = \sum_{k=0}^{\infty} \frac{\left\{\prod_{m=1}^{p} b(\kappa_m + a_m k;\vartheta_m + b_m k)\right\}}{\left\{\prod_{n=1}^{q} b(\widehat{\kappa}_n + c_n k;\widehat{\vartheta}_n + d_n k)\right\}} \frac{X^k}{k!},$$

where $m,n \in \mathbb{N}$, $k \in \mathbb{N}\cup\{0\}$, $\kappa_m,\vartheta_m,\widehat{\kappa}_n,\widehat{\vartheta}_n,X \in \mathbb{C}$, $p,q \in \mathbb{N}\cup\{0\}$, and $a_m,b_m,c_n,d_n \in \mathbb{R}_+$.
Now, we define the Wright generalized hypergeometric series as follows

$$\varphi_7^{\circledS}(X) := [{}_p\mathbb{W}_q]^n(X) = \sum_{s=0}^{n} \frac{\left\{\prod_{j=1}^{p}\Gamma(\kappa_j + a_j s)\right\}}{\left\{\prod_{k=1}^{q}\Gamma(\vartheta_k + b_k s)\right\}} \frac{X^s}{s!},$$

where $X,\kappa_j,\vartheta_k \in \mathbb{C}$, $s,j,k,q,p \in \mathbb{N}$, and $a_j,b_k \in \mathbb{R}_+$.

Let

$$\mathrm{diag}[\rho_1,\cdots,\rho_n]_{n\times n} = \begin{bmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \rho_n \end{bmatrix}_{n\times n}.$$

Note that $:= \mathrm{diag}[\rho_1,\cdots,\rho_n] \preceq \; := \mathrm{diag}[\rho_1,\cdots,\rho_n]$ if $\rho_i \leqslant \rho_i$ for each $1 \leqslant i \leqslant n$.

Assume the following matrix valued control function given by

$$\mathfrak{W}[X] = \mathrm{diag}\left[\varphi_1^{\circledS}(X),\cdots,\varphi_n^{\circledS}(X)\right]_{n\times n}.$$

Assume a mapping Υ from a vector space κ to normed linear space ϑ has Hyers-Ulam-Rassias stability. If we replace the control function of Hyers-Ulam-Rassias stability with $\mathfrak{W}[X]$, we say Υ has the multi-stability property.

In this paper, we investigate the multi-stability problem of ternary antiderivations associated to the following functional inequality in complex ternary Banach algebras through the fixed point method.

$$\begin{aligned}
&\mathrm{diag}\Big[\ \Big\|\mathscr{J}_1(\mathfrak{P},\alpha+\beta+\gamma) - \mathscr{J}_1(\mathfrak{P},\alpha+\gamma) - \mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma) - \mathscr{J}_1(\mathfrak{P},\alpha-\gamma)\Big\| \\
&\qquad ,\cdots, \Big\|\mathscr{J}_n(\mathfrak{P},\alpha+\beta+\gamma) - \mathscr{J}_n(\mathfrak{P},\alpha+\gamma) - \mathscr{J}_n(\mathfrak{P},\alpha+\beta-\gamma) - \mathscr{J}_n(\mathfrak{P},\alpha-\gamma)\Big\|\ \Big] \\
&\preceq \mathrm{diag}\Big[\ \Big\|\theta_1(\mathscr{J}_1(\mathfrak{P},\alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P},\alpha-\gamma) - \mathscr{J}_1(\mathfrak{P},\beta))\Big\| \\
&\qquad + \Big\|\theta_1'(\mathscr{J}_1(\mathfrak{P},\alpha-\gamma) + \mathscr{J}_1(\mathfrak{P},\alpha) - \mathscr{J}_1(\mathfrak{P},\gamma))\Big\|,\cdots, \\
&\qquad \Big\|\theta_n(\mathscr{J}_n(\mathfrak{P},\alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P},\alpha-\gamma) - \mathscr{J}_n(\mathfrak{P},\beta))\Big\| \\
&\qquad + \Big\|\theta_n'(\mathscr{J}_n(\mathfrak{P},\alpha-\gamma) + \mathscr{J}_n(\mathfrak{P},\alpha) - \mathscr{J}_n(\mathfrak{P},\gamma))\Big\|\ \Big],
\end{aligned} \tag{1.4}$$

for each $\alpha,\beta,\gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, where $\theta_1,\cdots,\theta_n$ and $\theta_1',\cdots,\theta_n'$ are fixed nonzero complex numbers with $|\theta_1| + |\theta_1'| < 2,\cdots,|\theta_n| + |\theta_n'| < 2$.

2. investigating the multi-stability and super-multi-stability associated to (1.4)

In this section, we investigate the concept of ternary antiderivation on ternary Banach algebras and introduce the super-multi-stability of ternary antiderivation in ternary Banach algebras, associated to the (1.4). For more details, see [2, 3, 4, 5, 6, 7].

Throughout this section, let $\mathscr{X}$ be a ternary Banach algebra. and $\theta := (\theta_1, \cdots, \theta_n)$ and $\theta' := (\theta'_1, \cdots, \theta'_n)$ are fixed nonzero complex numbers with $|\theta_1| + |\theta'_1| < 2, \cdots, |\theta_n| + |\theta'_n| < 2$.

A ternary Banach algebra is a complex Banach space $\mathscr{X}$, endowed with a ternary product $(\alpha, \beta, \gamma) \to [\alpha, \beta, \gamma]$ of $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ into $\mathscr{X}$, which $\mathbb{C}$-linear in the each variables, and associative in the sense that $[\alpha, \beta, [\gamma, \omega, \upsilon]] = [\alpha, [\omega, \gamma, \beta], \upsilon] = [[\alpha, \beta, \gamma], \omega, \upsilon]$, and satisfies $\|[\alpha, \beta, \gamma]\| \leqslant \|\alpha\| \cdot \|\beta\| \cdot \|\gamma\|$ for each $\alpha, \beta, \gamma, \omega, \upsilon \in \mathscr{X}$.

If a ternary Banach algebra $(\mathscr{X}, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $\rho \in \mathscr{X}$ such that $\alpha = [\alpha, \rho, \rho] = [\rho, \rho, \alpha]$ for each $\alpha \in \mathscr{X}$, then it is routine to verify that $\mathscr{X}$, endowed with $\alpha \circ \beta := [\alpha, \rho, \beta]$ and $\alpha^* := [\rho, \alpha, \rho]$, is a unital algebra. Conversely, if $(\mathscr{X}, \circ)$ is a unital algebra, then $[\alpha, \beta, \gamma] := \alpha \circ \beta^* \circ \gamma$ makes $\mathscr{X}$ into a ternary Banach algebra. If a ternary Banach algebra $(\mathscr{X}, [\cdot, \cdot, \cdot])$ has a unit, i.e., an element $\rho \in \mathscr{X}$ s.t. $\alpha = [\alpha, \rho, \rho] = [\rho, \rho, \alpha]$ for each $\alpha \in \mathscr{X}$, then $\mathscr{X}$ with the binary product $\alpha \circ \beta := [\alpha, \rho, \beta]$, is a usual Banach algebra.

A $\mathbb{C}$-linear mapping $\varpi : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$ is called a ternary homomorphism if $\varpi(\mathfrak{P}, [\alpha, \beta, \gamma]) = [\varpi(\mathfrak{P}, \alpha), \varpi(\mathfrak{P}, \beta), \varpi$ for each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. A $\mathbb{C}$-linear mapping $\varpi' : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$ is called a ternary derivation if $\varpi'(\mathfrak{P}, [\alpha, \beta, \gamma]) = [\varpi'(\mathfrak{P}, \alpha), \beta, \gamma] + [\alpha, \varpi'(\mathfrak{P}, \beta), \gamma] + [\alpha, \beta, \varpi'(\mathfrak{P}, \gamma)]$ for each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

2.1. Multi-stability of (θ, θ')-functional inequality (1.4)

In this subsection, we propose the multi-stability of the additive (θ, θ')-functional inequality (1.4) through the fixed point method.

Lemma 2.1. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying (1.4), for each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are additive.

Proof. Let $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, satisfies (1.4). Letting $\alpha = \beta = \gamma = 0$ in (1.4), we obtain $2\|\mathscr{J}_i(\mathfrak{P}, 0)\| \leqslant (|\theta_i| + |\theta'_i|)\|\mathscr{J}_i(\mathfrak{P}, 0)\|$, for $i = 1, \cdots, n$ and therefore $\mathscr{J}_i(\mathfrak{P}, 0) = 0$, since $|\theta_i| + |\theta'_i| < 2$. Putting $\gamma = \alpha$ in (1.4), we get $\|\mathscr{J}_i(\mathfrak{P}, 2\alpha + \beta) - \mathscr{J}_i(\mathfrak{P}, 2\alpha) - \mathscr{J}_i(\mathfrak{P}, \beta)\| \leqslant 0$ and so $\mathscr{J}_i(\mathfrak{P}, 2\alpha + \beta) = \mathscr{J}_i(\mathfrak{P}, 2\alpha) + \mathscr{J}_i(\mathfrak{P}, \beta)$ for each $\alpha, \beta \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Therefore $\mathscr{J}_i, (i = 1, \cdots, n)$, are additive. □

Theorem 2.2. Suppose $\underbrace{\varphi_{j_1}}_{1 \leqslant j_1 \leqslant n}, \cdots, \underbrace{\varphi_{j_n}}_{1 \leqslant j_n \leqslant n} : \mathscr{X}^3 \to [0, \infty)$ are functions s.t. there exist $(\mathcal{T}_1, \cdots, \mathcal{T}_n) \prec \overbrace{(1, \cdots, 1)}^{n}$ with

$$\begin{aligned} &\operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1 \leqslant j_1 \leqslant n}(\alpha, \beta, \gamma), \cdots, \underbrace{\varphi_{j_n}}_{1 \leqslant j_n \leqslant n}(\alpha, \beta, \gamma)\Big] \\ &\preceq \operatorname{diag}\Big[\frac{\mathcal{T}_1}{2}\underbrace{\varphi_{j_1}}_{1 \leqslant j_1 \leqslant n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathcal{T}_n}{2}\underbrace{\varphi_{j_n}}_{1 \leqslant j_n \leqslant n}(2\alpha, 2\beta, 2\gamma)\Big], \end{aligned} \tag{2.1}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\Big\| \mathscr{J}_1(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) \Big\| \\
&\quad , \cdots, \Big\| \mathscr{J}_n(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) \Big\| \Big] \\
&\preceq \operatorname{diag}\Big[\Big\| \theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta)) \Big\| \\
&\quad + \Big\| \theta'_1(\mathscr{J}_1(\mathfrak{P}, \alpha-z) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma)) \Big\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(\alpha, \beta, \gamma) \qquad (2.2) \\
&\quad , \cdots, \Big\| \theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta)) \Big\| \\
&\quad + \Big\| \theta'_n(\mathscr{J}_n(\mathfrak{P}, \alpha-z) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma)) \Big\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(\alpha, \beta, \gamma) \Big], \qquad (2.3)
\end{aligned}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Then there exist unique additive mappings $\mathscr{J}'_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, s.t.

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}'_n(\mathfrak{P}, \alpha)\| \Big] \\
&\preceq \operatorname{diag}\Big[\frac{\mathfrak{T}_1}{2(1-\mathfrak{T}_1)} \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \frac{\mathfrak{T}_n}{2(1-\mathfrak{T}_n)} \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right) \Big], \qquad (2.4)
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Proof. Letting $\alpha = \beta = \gamma = 0$ in (2.2), we have

$$\begin{aligned}
&\operatorname{diag}\Big[2\|\mathscr{J}_1(\mathfrak{P}, 0)\|, \cdots, 2\|\mathscr{J}_n(\mathfrak{P}, 0)\| \Big] \\
&\preceq \operatorname{diag}\Big[(|\theta_1| + |\theta'_1|)\|\mathscr{J}_1(\mathfrak{P}, 0)\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(0,0,0), \cdots, \\
&\qquad (|\theta_n| + |\theta'_n|)\|\mathscr{J}_n(\mathfrak{P}, 0)\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(0,0,0) \Big]
\end{aligned}$$

and thus $\mathscr{J}_i(\mathfrak{P}, 0) = 0, (i = 1, \cdots, n)$, since $|\theta_1| + |\theta'_1|, \cdots, |\theta_n| + |\theta'_n| < 2$ and by (2.1),

$$\underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(0,0,0), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(0,0,0) = 0.$$

Putting $\alpha = \gamma = \frac{\tau}{2}$ and $\beta = \tau$ in (2.2), we obtain

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, 2\tau) - 2\mathscr{J}_1(\mathfrak{P}, \tau)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, 2\tau) - 2\mathscr{J}_n(\mathfrak{P}, \tau)\| \Big] \qquad (2.5) \\
&\preceq \operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}\left(\frac{\tau}{2}, \tau, \frac{\tau}{2}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}\left(\frac{\tau}{2}, \tau, \frac{\tau}{2}\right) \Big],
\end{aligned}$$

for each $\tau \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Throughout this subsection, let $\hbar := (\hbar_1, \cdots, \hbar_n)$ and $\hbar' := (\hbar'_1, \cdots, \hbar'_n)$.

Now, consider the set

$$\nabla = \left\{ \hbar : \overbrace{(\mathscr{X}' \times \mathscr{X}) \times \cdots \times (\mathscr{X}' \times \mathscr{X})}^{n} \to \overbrace{\mathscr{X} \times \cdots \times \mathscr{X}}^{n} : \hbar(\mathfrak{P}, 0) = \overbrace{(0, \cdots, 0)}^{n} \right\}$$

and the mapping d defined on $\nabla \times \nabla$ by

$$\begin{aligned} d(\hbar, \hbar') = \inf \Bigg\{ & (\mu_1, \cdots, \mu_n) \in \mathbb{R}^n_+ : \\ & \operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P}, \alpha) - \hbar'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\hbar_n(\mathfrak{P}, \alpha) - \hbar'_n(\mathfrak{P}, \alpha)\| \ \Big] \\ & \preceq \operatorname{diag}\Big[\ \mu_1 \underbrace{j_1}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \mu_n \underbrace{j_n}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right) \ \Big], \forall \alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}' \Bigg\}, \end{aligned}$$

where as usual, $\inf \emptyset = (+\infty, \cdots, +\infty)$. d is a complete generalized metric on ∇.

Now, let us consider the linear mapping $\mathscr{L} := (\mathscr{L}_1, \cdots, \mathscr{L}_n) : \nabla \to \nabla$ such that $\mathscr{L}_i \hbar_i(\mathfrak{P}, \alpha) := 2\hbar_i\left(\mathfrak{P}, \frac{\alpha}{2}\right)$ for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Indeed, suppose $\hbar, \hbar' \in \nabla$ are given s.t. $d(\hbar, \hbar') = (\varepsilon_1, \cdots, \varepsilon_n)$, then

$$\begin{aligned} & \operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P}, \alpha) - \hbar'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\hbar_n(\mathfrak{P}, \alpha) - \hbar'_n(\mathfrak{P}, \alpha)\| \ \Big] \\ & \preceq \operatorname{diag}\Big[\ \varepsilon_1 \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \varepsilon_n \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right) \ \Big], \end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Hence

$$\begin{aligned} & \operatorname{diag}\Big[\ \|\mathscr{L}_1\hbar_1(\mathfrak{P}, \alpha) - \mathscr{L}_1\hbar'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{L}_n\hbar_n(\mathfrak{P}, \alpha) - \mathscr{L}_n\hbar'_n(\mathfrak{P}, \alpha)\| \ \Big] \\ & = \operatorname{diag}\Big[\ \left\|2\hbar_1\left(\mathfrak{P}, \frac{\alpha}{2}\right) - 2\hbar'_1\left(\mathfrak{P}, \frac{\alpha}{2}\right)\right\|, \cdots, \left\|2\hbar_n\left(\mathfrak{P}, \frac{\alpha}{2}\right) - 2\hbar'_n\left(\mathfrak{P}, \frac{\alpha}{2}\right)\right\| \ \Big] \\ & \preceq \operatorname{diag}\Big[\ 2\varepsilon_1 \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{4}, \frac{\alpha}{2}, \frac{\alpha}{4}\right), \cdots, 2\varepsilon_n \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{4}, \frac{\alpha}{2}, \frac{\alpha}{4}\right) \ \Big] \\ & \preceq \operatorname{diag}\Big[\ \mathcal{T}_1\varepsilon_1 \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \mathcal{T}_n\varepsilon_n \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right) \ \Big], \end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, that is $d(\hbar, \hbar') = (\varepsilon_1, \cdots, \varepsilon_n)$ implies that $d(\mathscr{L}\hbar(\mathfrak{P}, \alpha), \mathscr{L}\hbar'(\mathfrak{P}, \alpha)) \preceq (\mathcal{T}_1\varepsilon_1, \cdots, \mathcal{T}_n\varepsilon_n)$. This means that $d(\mathscr{L}\hbar(\mathfrak{P}, \alpha), \mathscr{L}\hbar'(\mathfrak{P}, \alpha)) \preceq (\mathcal{T}_1, \cdots, \mathcal{T}_n) d(\hbar, \hbar')$ for each $\hbar, \hbar' \in \nabla$.

Next, from (2.5), we get

$$\begin{aligned} & \operatorname{diag}\Big[\ \left\|\mathscr{J}_1(\mathfrak{P}, \alpha) - 2\mathscr{J}_1(\mathfrak{P}, \frac{\alpha}{2})\right\|, \cdots, \left\|\mathscr{J}_n(\mathfrak{P}, \alpha) - 2\mathscr{J}_n(\mathfrak{P}, \frac{\alpha}{2})\right\| \ \Big] \\ & \preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{4}, \frac{\alpha}{2}, \frac{\alpha}{4})\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{4}, \frac{\alpha}{2}, \frac{\alpha}{4})\right) \ \Big] \\ & \preceq \operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{2} \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \frac{\mathcal{T}_n}{2} \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right) \ \Big], \end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, it follows that $d(\mathscr{J}, \mathscr{L}\mathscr{J}) \preceq (\frac{\mathcal{T}_1}{2}, \cdots, \frac{\mathcal{T}_n}{2})$,in which $\mathscr{J} := (\mathscr{J}_1, \cdots, \mathscr{J}_n)$.

by the fixed point alternative we conclude the existence of unique fixed points of $\mathscr{L}_i$, $(i = 1, \cdots, n)$, that are, the existence of a mapping $\mathscr{J}'_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$ such that $\mathscr{J}'_i(\mathfrak{P}, \alpha) = 2\mathscr{J}'_i\left(\mathfrak{P}, \frac{\alpha}{2}\right)$, $\quad i = 1, \cdots, n$ with the following property: there exist $(\mu_1, \cdots, \mu_n) \in (0, \infty)^n$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}'_1(\mathfrak{P}, \alpha)\| \\
&\preceq \operatorname{diag}\Big[\ \mu_1 \underbrace{\varphi_{j_1}}_{1 \leqslant j_1 \leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \mu_n \underbrace{\varphi_{j_n}}_{1 \leqslant j_n \leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Since $\lim_{m\to\infty} d(\mathscr{L}^m\mathscr{J}, \mathscr{J}') = \underbrace{(0, \cdots, 0)}_{n}$,

$$\lim_{m\to\infty} 2^m \mathscr{J}\left(\mathfrak{P}, \frac{\alpha}{2^m}\right) = \mathscr{J}'(\mathfrak{P}, \alpha)$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$, and $\mathscr{J} := (\mathscr{J}_1, \cdots, \mathscr{J}_n)$, $\mathscr{J}' := (\mathscr{J}'_1, \cdots, \mathscr{J}'_n)$.

Also, $d(\mathscr{J}, \mathscr{J}') \preceq \overbrace{(\frac{1}{1-\mathcal{T}_1}, \cdots, \frac{1}{1-\mathcal{T}_1})}^{n}\, d(\mathscr{J}, \mathscr{L}\mathscr{J})$ which implies

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}'_1(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}'_n(\mathfrak{P}, \alpha)\|\ \Big] \\
&\preceq \frac{\mathcal{T}_1}{2(1-\mathcal{T}_1)} \underbrace{\varphi_{j_1}}_{1 \leqslant j_1 \leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \frac{\mathcal{T}_n}{2(1-\mathcal{T}_n)} \underbrace{\varphi_{j_n}}_{1 \leqslant j_n \leqslant n} \left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1'(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_1'(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_1'(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)\|, \cdots,\\
&\qquad \|\mathscr{J}_n'(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{J}_n'(\mathfrak{P},\alpha+\gamma)-\mathscr{J}_n'(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)\|\Big]\\
&= \operatorname{diag}\Big[\lim_{m\to\infty} 2^m \Big\| \mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\gamma}{2^n}\right)\\
&\qquad -\mathscr{J}_1\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)\Big\|, \cdots,\\
&\qquad \lim_{m\to\infty} 2^m \Big\| \mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m}\right)\\
&\qquad -\mathscr{J}_n\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)\Big\|\Big]\\
&\preceq \operatorname{diag}\Big[\lim_{m\to\infty} 2^m \Big\|\theta_1\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta}{2^m}\right)\right)\Big\|\\
&\qquad + \lim_{m\to\infty} 2^m \Big\|\theta_1'\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\gamma}{2^m}\right)\right)\Big\|\\
&\qquad + \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\mathfrak{P},\frac{\alpha}{2^m},\frac{\beta}{2^m},\frac{\gamma}{2^m}\right), \cdots,\\
&\qquad \lim_{m\to\infty} 2^m \Big\|\theta_n\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta}{2^m}\right)\right)\Big\|\\
&\qquad + \lim_{m\to\infty} 2^m \Big\|\theta_n'\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\gamma}{2^m}\right)\right)\Big\|\\
&\qquad + \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\mathfrak{P},\frac{\alpha}{2^m},\frac{\beta}{2^m},\frac{\gamma}{2^m}\right)\Big]\\
&= \operatorname{diag}\Big[\|\theta_1(\mathscr{J}_1'(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_1'(\mathfrak{P},\beta))\|\\
&\qquad + \|\theta_1'(\mathscr{J}_1'(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_1'(\mathfrak{P},\alpha)-\mathscr{J}_1'(\mathfrak{P},\gamma))\|, \cdots,\\
&\qquad \|\theta_n(\mathscr{J}_n'(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)-\mathscr{J}_n'(\mathfrak{P},\beta))\|\\
&\qquad + \|\theta_n'(\mathscr{J}_n'(\mathfrak{P},\alpha-\gamma)+\mathscr{J}_n'(\mathfrak{P},\alpha)-\mathscr{J}_n'(\mathfrak{P},\gamma))\|\Big],
\end{aligned}$$

for each $\alpha,\beta,\gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Therefore, by Lemma 2.1, the mappings $\mathscr{J}_i'$, $(i = 1, \cdots, n)$, are additive. □

2.2. Ternary antiderivations in ternary algebras

In this subsection, we propose the concept of ternary antiderivation in ternary Banach algebras and investigate the super-multi-stability of ternary antiderivations associated to (1.4) in ternary Banach algebras.

Definition 2.3. Let $\mathscr{X}$ be a ternary Banach algebra. $\mathbb{C}$-linear mappings $\mathscr{G}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}$, $(i = 1, \cdots, n)$, are called ternary antiderivations if they satisfy

$$[\mathscr{G}_i(\mathfrak{P},\alpha),\mathscr{G}_i(\mathfrak{P},\beta),\mathscr{G}_i(\mathfrak{P},\gamma)] = \mathscr{G}_i[\mathfrak{P},\mathscr{G}_i(\mathfrak{P},\alpha),\beta,\gamma]+\mathscr{G}_i[\mathfrak{P},\alpha,\mathscr{G}_i(\mathfrak{P},\beta),\gamma]+\mathscr{G}_i[\mathfrak{P},\alpha,\beta,\mathscr{G}_i(\mathfrak{P},\gamma)]$$

for each $\alpha,\beta,\gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$ and $i = 1, \cdots, n$.

Lemma 2.4. Assume $\mathscr{X}$ be a complex Banach algebra and assume $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, be additive mappings s.t. $\mathscr{J}_i(\mathfrak{P}, \mathfrak{J}\alpha) = \mathfrak{J}\mathscr{J}_i(\mathfrak{P}, \alpha), (i = 1, \cdots, n)$, for each $\mathfrak{J} \in \mathbb{T}^1 := \{\kappa \in \mathbb{C} : |\kappa| = 1\}$ and each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then $\mathscr{J}_i, (i = 1, \cdots, n)$, are $\mathbb{C}$-linear.

Theorem 2.5. Suppose $\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}, \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} : \mathscr{X}^3 \to [0, \infty)$ be functions. If there exist $(\mathcal{T}_1, \cdots, \mathcal{T}_n) < (1, \cdots, 1)$ with satisfying

$$\begin{aligned}
&\operatorname{diag}\Bigg[\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right) \Bigg] \\
&\preceq \operatorname{diag}\Bigg[\frac{\mathcal{T}_1}{8} \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathcal{T}_n}{8} \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(2\alpha, 2\beta, 2\gamma) \Bigg],
\end{aligned} \tag{2.6}$$

for ech $\mathfrak{J} \in \mathbb{T}^1$ and all $\alpha, \beta, \gamma \in \mathscr{X}$. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings s.t.

$$\begin{aligned}
&\operatorname{diag}\Bigg[\|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots, \\
&\qquad \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \Bigg] \\
&\preceq \operatorname{diag}\Bigg[\|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(\alpha, \beta, \gamma), \cdots, \\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(\alpha, \beta, \gamma) \Bigg],
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\operatorname{diag}\Bigg[\|[\mathscr{J}_1(\mathfrak{P}, \alpha), \mathscr{J}_1(\mathfrak{P}, \beta), \mathscr{J}_1(\mathfrak{P}, \gamma)] - \mathscr{J}_1[\mathfrak{P}, \mathscr{J}_1(\mathfrak{P}, \alpha), \beta, \gamma] \\
&\qquad + \mathscr{J}_1[\mathfrak{P}, \alpha, \mathscr{J}_1(\mathfrak{P}, \beta), \gamma] + \mathscr{J}_1[\mathfrak{P}, \alpha, \beta, \mathscr{J}_1(\mathfrak{P}, \gamma)]\|, \cdots, \\
&\qquad \|[\mathscr{J}_n(\mathfrak{P}, \alpha), \mathscr{J}_n(\mathfrak{P}, \beta), \mathscr{J}_n(\mathfrak{P}, \gamma)] - \mathscr{J}_n[\mathfrak{P}, \mathscr{J}_n(\mathfrak{P}, \alpha), \beta, \gamma] \\
&\qquad + \mathscr{J}_n[\mathfrak{P}, \alpha, \mathscr{J}_n(\mathfrak{P}, \beta), \gamma] + \mathscr{J}_n[\mathfrak{P}, \alpha, \beta, \mathscr{J}_n(\mathfrak{P}, \gamma)]\| \Bigg] \\
&\preceq \operatorname{diag}\Bigg[\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(\alpha, \beta, \gamma), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(\alpha, \beta, \gamma) \Bigg],
\end{aligned} \tag{2.8}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. If $\mathscr{J}_i, (i = 1, \cdots, n)$, are continuous and $\mathscr{J}_i(\mathfrak{P}, 2\alpha) = 2\mathscr{J}_i(\mathfrak{P}, \alpha)$ for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are ternary antiderivations.

Proof. Let $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, satisfie (2.7).

Letting $\mathfrak{J} = 1$ and $\alpha = \beta = \gamma = 0$ in (2.7), we get

$$\begin{aligned}
&\operatorname{diag}\Big[\ 2\|\mathscr{J}_1(\mathfrak{P}, 0)\|, \cdots, 2\|\mathscr{J}_n(\mathfrak{P}, 0)\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ (|\theta_1| + |\theta_1'|)\|\mathscr{J}_1(\mathfrak{P}, 0)\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(0,0,0)\ , \cdots, \\
&\qquad (|\theta_n| + |\theta_n'|)\|\mathscr{J}_n(\mathfrak{P}, 0)\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_1\leqslant n}(0,0,0)\ \Big],
\end{aligned}$$

and thus $\mathscr{J}_i(\mathfrak{P}, 0) = 0, (i = 1, \cdots, n)$, since $|\theta_1| + |\theta_1'|, \cdots, |\theta_n| + |\theta_n'| < 2$ and by (2.6),

$$\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(0,0,0), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(0,0,0) = 0.$$

Putting $\alpha = \gamma = \frac{\tau}{2}$ and $\gamma = \tau$ in (2.7), we obtain

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, 2\mathfrak{J}\tau) - 2\mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \tau)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, 2\mathfrak{J}\tau) - 2\mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \tau)\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\tau}{2}, \tau, \frac{\tau}{2}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\tau}{2}, \tau, \frac{\tau}{2}\right)\ \Big],
\end{aligned} \tag{2.9}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\tau \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Let $\hbar := (\hbar_1, \cdots, \hbar_n)$, and $\hbar' := (\hbar_1', \cdots, \hbar_n')$.

Next, consider the set

$$\nabla := \{\hbar : (\mathscr{X}' \times \mathscr{X})^n \to \mathscr{X}^n : \hbar(\mathfrak{P}, 0) = \overbrace{(0, \cdots, 0)}^{n}\}$$

and define the generalized metric on ∇

$$\begin{aligned}
d(\hbar, \hbar') = \inf\Big\{&(\mu_1, \cdots, \mu_n) \in \mathbb{R}^n_{\geqslant 0} : \\
&\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P}, \alpha) - \hbar_1'(\mathfrak{P}, \alpha)\|, \cdots, \|\hbar_n(\mathfrak{P}, \alpha) - \hbar_n'(\mathfrak{P}, \alpha)\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ \mu_1 \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \mu_n \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right)\ \Big], \ \forall \alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'\Big\},
\end{aligned}$$

where as usual, $\inf(\emptyset, \cdots, \emptyset) = (+\infty, \cdots, +\infty)$. It is easy to show that (∇, d) is a complete generalized metric space.

Let $\mathscr{L} := (\mathscr{L}_1, \cdots, \mathscr{L}_n)$. Now we define the linear mapping $\mathscr{L} : \nabla \to \nabla$ s.t.

$$\mathscr{L}_i\hbar_i(\mathfrak{P}, \alpha) = 2\mathfrak{J}\hbar_i\left(\mathfrak{P}, \frac{\alpha}{2\mathfrak{J}}\right), \ \forall i = 1, \cdots, n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Assume $\hbar, \hbar' \in \nabla$ be given s.t. $d(\hbar, \hbar') = (\varepsilon_1, \cdots, \varepsilon_n)$. Then

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\hbar_1(\mathfrak{P}, \alpha) - \hbar_1'(\mathfrak{P}, \alpha)\|, \cdots, \|\hbar_n(\mathfrak{P}, \alpha) - \hbar_n'(\mathfrak{P}, \alpha)\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ \varepsilon_1 \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right), \cdots, \varepsilon_n \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2}, \alpha, \frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Hence

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{L}_1\hbar_1(\mathfrak{P},\alpha)-\mathscr{L}_1\hbar_1'(\mathfrak{P},\alpha)\|,\cdots,\|\mathscr{L}_n\hbar_n(\mathfrak{P},\alpha)-\mathscr{L}_n\hbar_n'(\mathfrak{P},\alpha)\|\ \Big]\\
&=\operatorname{diag}\Big[\ \left\|2\mathfrak{J}\hbar_1\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)-2\mathfrak{J}\hbar_1'\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|,\cdots,\left\|2\mathfrak{J}\hbar_n\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)-2\mathfrak{J}\hbar_n'\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ 2\varepsilon_1\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right),\cdots,2\varepsilon_n\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right)\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \frac{\mathcal{T}_1}{4}\varepsilon_1\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathcal{T}_n}{4}\varepsilon_n\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. So $d(\hbar,\hbar')=(\varepsilon_1,\cdots,\varepsilon_n)$ implies that $d(\mathscr{L}\hbar(\mathfrak{P},\alpha),\mathscr{L}\hbar'(\mathfrak{P},\alpha))\preceq(\frac{\mathcal{T}_1}{4}\varepsilon_1,\cdots,\frac{\mathcal{T}_n}{4}\varepsilon_n)$. Hence

$$d(\mathscr{L}\hbar(\mathfrak{P},\alpha),\mathscr{L}\hbar'(\mathfrak{P},\alpha))\preceq(\frac{\mathcal{T}_1}{4},\cdots,\frac{\mathcal{T}_n}{4})d(\hbar,\hbar')$$

for each $\hbar,\hbar'\in\nabla$. According to (2.9),

$$\begin{aligned}
&\operatorname{diag}\left[\left\|\mathscr{J}_1(\mathfrak{P},\alpha)-2\mathfrak{J}\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|,\cdots,\left\|\mathscr{J}_n(\mathfrak{P},\alpha)-2\mathfrak{J}\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right)\right\|\ \right]\\
&\preceq\operatorname{diag}\left[\ \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right),\cdots,\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{4\mathfrak{J}},\frac{\alpha}{2\mathfrak{J}},\frac{\alpha}{4\mathfrak{J}}\right)\ \right]\\
&\preceq\operatorname{diag}\ [\frac{\mathcal{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathcal{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$ and so $d(\mathscr{J},\mathscr{L}\mathscr{J})\preceq(\frac{\mathcal{T}_1}{8},\cdots,\frac{\mathcal{T}_n}{8})$.

By the fixed point alternative we deduce the existence of unique fixed points of $\mathscr{L}_i,(i=1,\cdots,n)$, that is, the existence of mappings $\mathscr{G}_i:\mathscr{X}'\times\mathscr{X}\to\mathscr{X},(i=1,\cdots,n)$, s.t.

$$\mathscr{G}_i(\mathfrak{P},\alpha)=2\mathfrak{J}\mathscr{G}_i\left(\mathfrak{P},\frac{\alpha}{2\mathfrak{J}}\right),\ \forall i=1,\cdots,n$$

with the following property: there exist $\mathcal{T}_1,\cdots,\mathcal{T}_n\in(0,\infty)$ satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\alpha)\|,\cdots,\|\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\alpha)\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \mathcal{T}_1\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\mathcal{T}_n\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \Big],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Since $\lim_{m\to\infty}d(\mathscr{L}^m\mathscr{J},\mathscr{G})=(0,\cdots,0)$, in which $\mathscr{J}:=(\mathscr{J}_1,\cdots,\mathscr{J}_n)$, $\mathscr{G}:=(\mathscr{G}_1,\cdots,\mathscr{G}_n)$, and $\mathscr{L}:=(\mathscr{L}_1,\cdots,\mathscr{L}_n)$, then

$$\lim_{m\to\infty}2^m\mathfrak{J}^m\mathscr{J}_i\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)=\mathscr{G}_i(\mathfrak{P},\alpha),\quad\forall i=1,\cdots,n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. In particular,

$$\mathscr{G}_i(\mathfrak{P},\alpha)=\lim_{m\to\infty}2^m\mathscr{J}_i\left(\mathfrak{P},\frac{\alpha}{2^m}\right)=\mathscr{J}_i(\mathfrak{P},\alpha),\quad\forall i=1,\cdots,n$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, since $\mathscr{J}_i(\mathfrak{P}, 2\alpha) = 2\mathscr{J}_i(\mathfrak{P}, \alpha), (i = 1, \cdots, n)$, for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.
Also, $d(\mathscr{J}, \mathscr{G}) \preceq (\frac{1}{1-\frac{\mathfrak{T}_1}{4}}, \cdots, \frac{1}{1-\frac{\mathfrak{T}_n}{4}})d(\mathscr{J}, \mathscr{L}\mathscr{J})$ which implies

$$\begin{aligned}
&\operatorname{diag}\left[\ \|\mathscr{J}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\alpha)\|\ ,\cdots,\ \|\mathscr{J}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\alpha)\|\ \right]\\
&\preceq \operatorname{diag}\left[\ \frac{\mathfrak{T}_1}{2(4-\mathfrak{T}_1)}\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right),\cdots,\frac{\mathfrak{T}_n}{2(4-\mathfrak{T}_n)}\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2},\alpha,\frac{\alpha}{2}\right)\ \right],
\end{aligned}$$

for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. It follows from (2.6) and (2.7) that

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{G}_1(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{G}_1(\mathfrak{P},\alpha+\gamma)-\mathscr{G}_1(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)\|\\
&\qquad,\cdots,\ \|\mathscr{G}_n(\mathfrak{P},\alpha+\beta+\gamma)-\mathscr{G}_n(\mathfrak{P},\alpha+\gamma)-\mathscr{G}_n(\mathfrak{P},\beta-\alpha+\gamma)-\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)\|\ \Big]\\
&=\operatorname{diag}\Big[\ \lim_{m\to\infty}\Big\|2^m\mathfrak{J}^m\Big(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m\mathfrak{J}^m}\right)\\
&\qquad\qquad-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)\Big)\Big\|\\
&\qquad,\cdots,\ \lim_{m\to\infty}\Big\|2^m\mathfrak{J}^m\Big(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\gamma}{2^m\mathfrak{J}^m}\right)\\
&\qquad\qquad-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta-\alpha+\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)\Big)\Big\|\ \Big]\\
&\preceq\operatorname{diag}\Big[\ \lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_1\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\qquad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_1'\left(\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_1\left(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\qquad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\\
&\qquad,\cdots,\ \lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_n\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha+\beta-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\beta}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\qquad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\Big\|\theta_n'\left(\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha-\gamma}{2^m\mathfrak{J}^m}\right)+\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)-\mathscr{J}_n\left(\mathfrak{P},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\right)\Big\|\\
&\qquad+\lim_{m\to\infty}2^m|\mathfrak{J}|^m\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2^m\mathfrak{J}^m},\frac{\beta}{2^m\mathfrak{J}^m},\frac{\gamma}{2^m\mathfrak{J}^m}\right)\ \Big]\\
&=\operatorname{diag}\Big[\ \|\theta_1(\mathscr{G}_1(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)-\mathscr{G}_1(\mathfrak{P},\beta))\|\\
&\qquad\qquad+\|\theta_1'(\mathscr{G}_1(\mathfrak{P},\alpha-\gamma)+\mathscr{G}_1(\mathfrak{P},\alpha)-\mathscr{G}_1(\mathfrak{P},\gamma))\|\\
&\qquad,\cdots,\ \|\theta_n(\mathscr{G}_n(\mathfrak{P},\alpha+\beta-\gamma)+\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)-\mathscr{G}_n(\mathfrak{P},\beta))\|\\
&\qquad\qquad+\|\theta_n'(\mathscr{G}_n(\mathfrak{P},\alpha-\gamma)+\mathscr{G}_n(\mathfrak{P},\alpha)-\mathscr{G}_n(\mathfrak{P},\gamma))\|\ \Big],
\end{aligned}$$

for each $\alpha, \beta, \gamma \in \mathscr{X},\ \mathfrak{P} \in \mathscr{X}'$. According to Lemma 2.1, the mappings $\mathscr{G}_i, (i = 1, \cdots, n)$, are additive.

Putting $\alpha = \gamma = \frac{\tau}{2}$ and $\beta = 0$ in (2.7), we have

$$\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P},\mathfrak{J}\tau) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P},\tau)\|\ ,\cdots,\ \|\mathscr{J}_n(\mathfrak{P},\mathfrak{J}\tau) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P},\tau)\|\ \Big]$$
$$\preceq \operatorname{diag}\Big[\ \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\tau}{2},0,\frac{\tau}{2}\right)\ ,\cdots,\ \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\tau}{2},0,\frac{\tau}{2}\right)\ \Big],$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\tau \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Thus

$$\operatorname{diag}\Big[\ \|\mathscr{G}_1(\mathfrak{P},\mathfrak{J}\alpha) - \mathfrak{J}\mathscr{G}_1(\mathfrak{P},\alpha)\|,\cdots,\|\mathscr{G}_n(\mathfrak{P},\mathfrak{J}\alpha) - \mathfrak{J}\mathscr{G}_n(\mathfrak{P},\alpha)\|\ \Big]$$
$$= \operatorname{diag}\Big[\ \lim_{m\to\infty} 2^m|\mathfrak{J}|^m \left\|\mathscr{J}_1\left(\mathfrak{P},\mathfrak{J}\frac{\alpha}{2^m\mathfrak{J}^m}\right) - \mathfrak{J}\mathscr{J}_1\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)\right\|$$
$$,\cdots,\ \lim_{m\to\infty} 2^m|\mathfrak{J}|^m \left\|\mathscr{J}_n\left(\mathfrak{P},\mathfrak{J}\frac{\alpha}{2^m\mathfrak{J}^m}\right) - \mathfrak{J}\mathscr{J}_n\left(\mathfrak{P},\frac{\alpha}{2^m\mathfrak{J}^m}\right)\right\|\ \Big]$$
$$\preceq \operatorname{diag}\Big[\ \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2^{m+1}\mathfrak{J}^m},0,\frac{\alpha}{2^{m+1}\mu^m}\right)$$
$$,\cdots,\ \lim_{m\to\infty} 2^m \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2^{m+1}\mathfrak{J}^m},0,\frac{\alpha}{2^{m+1}\mu^m}\right)\ \Big]$$
$$\preceq \operatorname{diag}\Big[\ \lim_{m\to\infty}\left(\frac{\mathfrak{T}_1}{4}\right)^m \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{2},0,\frac{\alpha}{2}\right),\cdots,\lim_{m\to\infty}\left(\frac{\mathfrak{T}_n}{4}\right)^m \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{2},0,\frac{\alpha}{2}\right)\ \Big],$$

which tend to zero as $n \to \infty$ and so $\mathscr{G}_i(\mathfrak{P},\mathfrak{J}\alpha) = \mathfrak{J}\mathscr{G}_i(\mathfrak{P},\alpha), (i = 1,\dots,n)$, for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Therefore, according to Lemma 2.4, the mappings $\mathscr{G}_i, (i = 1,\dots,n)$, are $\mathbb{C}$-linear.

Since $\mathscr{J}_i = \mathscr{G}_i, (i = 1, \dots, n)$, is continuous and $\mathbb{C}$-linear, it follows from (2.6) and (2.8) that

$$\begin{aligned}
\operatorname{diag}\Big[\quad & \Big\| [\mathscr{G}_1(\mathfrak{P}, \alpha), \mathscr{G}_1(\mathfrak{P}, \beta), \mathscr{G}_1(\mathfrak{P}, \gamma)] - \mathscr{G}_1(\mathfrak{P}, [\mathscr{G}_1(\mathfrak{P}, \alpha), \beta, \gamma]) \\
& \quad - \mathscr{G}_1(\mathfrak{P}, [\alpha, \mathscr{G}_1(\mathfrak{P}, \beta), \gamma]) - \mathscr{G}_1(\mathfrak{P}, [\alpha, \beta, \mathscr{G}_1(\mathfrak{P}, \gamma)]) \Big\| , \cdots , \\
& \Big\| [\mathscr{G}_n(\mathfrak{P}, \alpha), \mathscr{G}_n(\mathfrak{P}, \beta), \mathscr{G}_n(\mathfrak{P}, \gamma)] - \mathscr{G}_n(\mathfrak{P}, [\mathscr{G}_n(\mathfrak{P}, \alpha), \beta, \gamma]) \\
& \quad - \mathscr{G}_n(\mathfrak{P}, [\alpha, \mathscr{G}_n(\mathfrak{P}, \beta), \gamma]) - \mathscr{G}_n(\mathfrak{P}, [\alpha, \beta, \mathscr{G}_n(\mathfrak{P}, \gamma)]) \Big\| \quad \Big] \\
= \operatorname{diag}\Big[\quad & \lim_{m\to\infty} \Big\| 2^{3m}\mathfrak{J}^{3m} \Big[\mathscr{J}_1\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_1\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_1\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big] \\
& - 2^m\mathfrak{J}^m \mathscr{G}_1\Big(\mathfrak{P}, \Big[\mathscr{J}_1\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \beta, \gamma \Big]\Big) - 2^m\mathfrak{J}^m \mathscr{G}_1\Big(\mathfrak{P}, \Big[\alpha, \mathscr{J}_1\Big(\frac{\beta}{2^m\mathfrak{J}^m}\Big), \gamma \Big]\Big) \\
& - 2^m\mathfrak{J}^m \mathscr{G}_1\Big(\mathfrak{P}, \Big[\alpha, \beta, \mathscr{J}_1\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big]\Big) \Big\| \ , \cdots , \\
& \lim_{m\to\infty} \Big\| 2^{3m}\mathfrak{J}^{3m} \Big[\mathscr{J}_n\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_n\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_n\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big] \\
& - 2^m\mathfrak{J}^m \mathscr{G}_n\Big(\mathfrak{P}, \Big[\mathscr{J}_n\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \beta, \gamma \Big]\Big) - 2^m\mathfrak{J}^m \mathscr{G}_n\Big(\mathfrak{P}, \Big[\alpha, \mathscr{J}_1\Big(\frac{\beta}{2^m\mathfrak{J}^m}\Big), \gamma \Big]\Big) \\
& - 2^m\mathfrak{J}^m \mathscr{G}_n\Big(\mathfrak{P}, \Big[\alpha, \beta, \mathscr{J}_n\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big]\Big) \Big\| \quad \Big] \\
= \operatorname{diag}\Big[\quad & \lim_{m\to\infty} 2^{3m} \Big\| \Big[\mathscr{J}_1\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_1\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_1\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big] \\
& - \mathscr{J}_1\Big(\mathfrak{P}, \Big[\mathscr{J}_1\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \frac{\beta}{2^m\mathfrak{J}^m}, \frac{\gamma}{2^m\mathfrak{J}^m} \Big]\Big) - \mathscr{J}_1\Big(\mathfrak{P}, \Big[\frac{\alpha}{2^m\mathfrak{J}^m}, \mathscr{J}_1\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \frac{\gamma}{2^m\mathfrak{J}^m} \Big]\Big) \\
& - \mathscr{J}_1\Big(\mathfrak{P}, \Big[\frac{\alpha}{2^m\mathfrak{J}^m}, \frac{\beta}{2^m\mathfrak{J}^m}, \mathcal{F}\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big]\Big) \Big\| \quad , \cdots , \\
& \lim_{m\to\infty} 2^{3m} \Big\| \Big[\mathscr{J}_n\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_n\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \mathscr{J}_n\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big] \\
& - \mathscr{J}_n\Big(\mathfrak{P}, \Big[\mathscr{J}_n\Big(\mathfrak{P}, \frac{\alpha}{2^m\mathfrak{J}^m}\Big), \frac{\beta}{2^m\mathfrak{J}^m}, \frac{\gamma}{2^m\mathfrak{J}^m} \Big]\Big) - \mathscr{J}_n\Big(\mathfrak{P}, \Big[\frac{\alpha}{2^m\mathfrak{J}^m}, \mathscr{J}_n\Big(\mathfrak{P}, \frac{\beta}{2^m\mathfrak{J}^m}\Big), \frac{\gamma}{2^m\mathfrak{J}^m} \Big]\Big) \\
& - \mathscr{J}_n\Big(\mathfrak{P}, \Big[\frac{\alpha}{2^m\mathfrak{J}^m}, \frac{\beta}{2^m\mathfrak{J}^m}, \mathscr{J}_n\Big(\mathfrak{P}, \frac{\gamma}{2^m\mathfrak{J}^m}\Big) \Big]\Big) \Big\| \quad \Big] \\
\preceq \operatorname{diag}\Big[\quad & \lim_{m\to\infty} 2^{3m} \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} \Big(\frac{\alpha}{2^m\mathfrak{J}^m}, \frac{\beta}{2^m\mathfrak{J}^m}, \frac{\gamma}{2^m\mathfrak{J}^m} \Big), \cdots , \lim_{m\to\infty} 2^{3m} \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} \Big(\frac{\alpha}{2^m\mathfrak{J}^m}, \frac{\beta}{2^m\mathfrak{J}^m}, \frac{\gamma}{2^m\mathfrak{J}^m} \Big) \ \Big] \\
\preceq \operatorname{diag}\Big[\quad & \lim_{m\to\infty} \mathcal{T}_1^m \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n} (\alpha, \beta, \gamma), \cdots , \lim_{m\to\infty} \mathcal{T}_n^m \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} (\alpha, \beta, \gamma) \quad \Big],
\end{aligned}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Since $(\mathcal{T}_1, \cdots, \mathcal{T}_n) < (1, \cdots, 1)$, the $\mathbb{C}$-linear mappings $\mathscr{G}_i, (i = 1, \cdots, n)$, are ternary antiderivations. Thus the mappings $\mathscr{J}_i, (i = 1, \cdots, n)$, are ternary antiderivations. □

2.3. Super-multi-stability of continuous ternary antiderivations in ternary Banach algebras

In this subsection, we investigate the super-multi-stability of continuous ternary antiderivations in ternary Banach algebras.

Theorem 2.6. Let $\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}, \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n} : \mathscr{X}^3 \to [0,\infty)$ be functions. If there exist $(\mathfrak{T}_1, \cdots, \mathfrak{T}_n) < (1, \cdots, 1)$ with satisfying

$$\begin{aligned}
&\operatorname{diag}\left[\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right)\right] \\
&\preceq \operatorname{diag}\left[\frac{\mathfrak{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathfrak{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(2\alpha, 2\beta, 2\gamma)\right],
\end{aligned} \tag{2.10}$$

for each $\mathfrak{J}$ with $|\mathfrak{J}| < 1$ (resp. $|\mathfrak{J}| > 1$) and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. Let $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, be mappings satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\ \|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\| \\
&\quad, \cdots, \ \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\|\ \Big] \\
&\preceq \operatorname{diag}\Big[\ \|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(\alpha, \beta, \gamma) \\
&\quad, \cdots, \ \|\theta_n(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(\alpha, \beta, \gamma)\ \Big],
\end{aligned} \tag{2.11}$$

and (2.8). If $\mathscr{J}_i, (i = 1, \cdots, n)$, are continuous and $\mathscr{J}_i(\mathfrak{P}, 2\alpha) = 2\mathscr{J}_i(\mathfrak{P}, \alpha)$ for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are ternary antiderivations.

Proof. Suppose $\mathfrak{J} \in \mathbb{T}^1$. Then there exists a sequence $\{\mathfrak{J}_m\}_{m=1}^{\infty}$ with $|\mathfrak{J}_m| < 1$ (resp. $|\mathfrak{J}_m| > 1$) s.t.

$$\lim_{m\to\infty} \mathfrak{J}_m = \mathfrak{J}.$$

By (2.10) and (2.11) we get

$$\begin{aligned}
&\operatorname{diag}\left[\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}\left(\frac{\alpha}{\mathfrak{J}_m}, \frac{\beta}{\mathfrak{J}_m}, \frac{\gamma}{\mathfrak{J}_m}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}\left(\frac{\alpha}{\mathfrak{J}_m}, \frac{\beta}{\mathfrak{J}_m}, \frac{\gamma}{\mathfrak{J}_m}\right)\right] \\
&\preceq \operatorname{diag}\left[\frac{\mathfrak{T}_1}{8}\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathfrak{T}_n}{8}\underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(2\alpha, 2\beta, 2\gamma)\right]
\end{aligned}$$

for each $\mathfrak{J}_m$ with $|\mathfrak{J}_m| < 1$ (resp. $|\mathfrak{J}_m| > 1$) and each $\alpha, \beta, \gamma \in \mathscr{X}$, and

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}_m(\alpha+\beta+\gamma)) - \mathfrak{J}_m \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}_m \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}_m \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\| \\
&\quad , \cdots, \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}_m(\alpha+\beta+\gamma)) - \mathfrak{J}_m \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J}_m \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J}_m \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \Big] \\
&\preceq \operatorname{diag}\Big[\|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(\alpha, \beta, \gamma) \\
&\qquad , \cdots, \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(\alpha, \beta, \gamma) \Big],
\end{aligned}$$

for each positive integers m and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Passing to the limit as $n \to \infty$, and using the continuity of $\underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}, \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}, \mathscr{J}_1, \cdots, \mathscr{J}_n$ and $\|\cdot\|$, we obtain

$$\begin{aligned}
&\operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}\left(\frac{\alpha}{\mathfrak{J}}, \frac{\beta}{\mathfrak{J}}, \frac{\gamma}{\mathfrak{J}}\right) \Big] \\
&\preceq \operatorname{diag}\Big[\frac{\mathcal{T}_1}{8} \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(2\alpha, 2\beta, 2\gamma), \cdots, \frac{\mathcal{T}_n}{8} \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(2\alpha, 2\beta, 2\gamma) \Big],
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J} \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J} \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J} \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\| \\
&\quad , \cdots, \|\mathscr{J}_n(\mathfrak{P}, \mathfrak{J}(\alpha+\beta+\gamma)) - \mathfrak{J} \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{J} \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{J} \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \Big] \\
&\preceq \operatorname{diag}\Big[\|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}}_{1\leqslant j_1 \leqslant n}(\alpha, \beta, \gamma) \\
&\qquad , \cdots, \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}}_{1\leqslant j_n \leqslant n}(\alpha, \beta, \gamma) \Big]
\end{aligned}$$

for each $\mathfrak{J} \in \mathbb{T}^1$ and each $\alpha, \beta, \gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$.

Thus, by the same reasoning as in the proof of Theorem 2.5, the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are ternary antiderivations. □

2.4. Application

Here, let $n = 7$.

Corollary 2.7. Suppose $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, are mappings satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots, \\
&\qquad \|\mathscr{J}_n(\mathfrak{P}, \alpha+\beta+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \Big] \\
&\preceq \operatorname{diag}\Big[\|\theta_1(\mathscr{J}_1(\alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}^{Ⓢ}}_{1\leqslant j_1\leqslant n}(\|[\alpha, \beta, \gamma]\|), \cdots, \\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}^{Ⓢ}}_{1\leqslant j_n\leqslant n}(\|[\alpha, \beta, \gamma]\|) \Big],
\end{aligned}$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$. Then there exists unique additive mappings $\mathscr{J}_i' : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, s.t.

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1'(\mathfrak{P}, \alpha)\|, \cdots, \|\mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n'(\mathfrak{P}, \alpha)\| \Big] \\
&\preceq \operatorname{diag}\Big[\underbrace{\varphi_{j_1}^{Ⓢ}}_{1\leqslant j_1\leqslant n}(\|[\alpha, \alpha, \alpha]\|), \cdots, \underbrace{\varphi_{j_n}^{Ⓢ}}_{1\leqslant j_n\leqslant n}(\|[\alpha, \alpha, \alpha]\|) \Big], \quad \forall i = 1, \cdots, n
\end{aligned}$$

for each $\alpha \in \mathscr{X}$, $\mathfrak{P} \in \mathscr{X}'$.

Proof. The proof follows from Theorem 2.2 by letting $\mathfrak{T}_i = \frac{2}{3}, (i = 1, \cdots, n)$, and

$$\operatorname{diag}\Big[\underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(\alpha, \beta, \gamma), \cdots, \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(\alpha, \beta, \gamma) \Big] := \operatorname{diag}\Big[\underbrace{\varphi_{j_1}^{Ⓢ}}_{1\leqslant j_1\leqslant n}(\|[\alpha, \beta, \gamma]\|), \cdots, \underbrace{\varphi_{j_n}^{Ⓢ}}_{1\leqslant j_n\leqslant n}(\|[\alpha, \beta, \gamma]\|) \Big],$$

for each $\alpha, \beta, \gamma \in \mathscr{X}$. □

Corollary 2.8. Assume $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1, \cdots, n)$, be mappings satisfying

$$\begin{aligned}
&\operatorname{diag}\Big[\|\mathscr{J}_1(\mathfrak{I}(\mathfrak{P}, \alpha+\beta+\gamma)) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma)\|, \cdots, \\
&\qquad \|\mathscr{J}_n(\mathfrak{I}(\mathfrak{P}, \alpha+\beta+\gamma)) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \alpha+\gamma) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \beta-\alpha+\gamma) - \mathfrak{I}\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma)\| \Big] \\
&\preceq \operatorname{diag}\Big[\|\theta_1(\mathscr{J}_1(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_1(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_1'(\mathscr{J}_1(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_1(\mathfrak{P}, \alpha) - \mathscr{J}_1(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_1}^{Ⓢ}}_{1\leqslant j_1\leqslant n}\Big(\|[[\alpha, \alpha, \beta], \beta, \gamma]\|\Big), \cdots, \\
&\qquad \|\theta_n(\mathscr{J}_n(\mathfrak{P}, \alpha+\beta-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) - \mathscr{J}_n(\mathfrak{P}, \beta))\| \\
&\qquad + \|\theta_n'(\mathscr{J}_n(\mathfrak{P}, \alpha-\gamma) + \mathscr{J}_n(\mathfrak{P}, \alpha) - \mathscr{J}_n(\mathfrak{P}, \gamma))\| + \underbrace{\varphi_{j_n}^{Ⓢ}}_{1\leqslant j_n\leqslant n}\Big(\|[[\alpha, \alpha, \beta], \beta, \gamma]\|\Big) \Big],
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{diag}\Bigg[\ \Big\| [\mathscr{J}_1(\mathfrak{P},\alpha),\mathscr{J}_1(\mathfrak{P},\beta),\mathscr{J}_1(\mathfrak{P},\gamma)] - \mathscr{J}_1\Big(\mathfrak{P},[\mathscr{J}_1(\mathfrak{P},\alpha),\beta,\gamma]\Big) \\
&\qquad + \mathscr{J}_1\Big(\mathfrak{P},[\alpha,\mathscr{J}_1(\mathfrak{P},\beta),\gamma]\Big) + \mathscr{J}_1\Big(\mathfrak{P},[\alpha,\beta,\mathscr{J}_1(\mathfrak{P},\gamma)]\Big)\Big\| \\
&\quad ,\cdots,\ \Big\| [\mathscr{J}_n(\mathfrak{P},\alpha),\mathscr{J}_n(\mathfrak{P},\beta),\mathscr{J}_n(\mathfrak{P},\gamma)] - \mathscr{J}_n\Big(\mathfrak{P},[\mathscr{J}_n(\mathfrak{P},\alpha),\beta,\gamma]\Big) \\
&\qquad + \mathscr{J}_n\Big(\mathfrak{P},[\alpha,\mathscr{J}_n(\mathfrak{P},\beta),\gamma]\Big) + \mathscr{J}_n\Big(\mathfrak{P},[\alpha,\beta,\mathscr{J}_n(\mathfrak{P},\gamma)]\Big)\Big\|\ \Bigg] \\
&\preceq \operatorname{diag}\Bigg[\ \underbrace{\varphi^{Ⓢ}_{j_1}}_{1\leqslant j_1\leqslant n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ ,\cdots,\ \underbrace{\varphi^{Ⓢ}_{j_n}}_{1\leqslant j_n\leqslant n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ \Bigg],
\end{aligned}$$

for each $\alpha,\beta,\gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. If $\mathscr{J}_i, (i = 1,\cdots,n)$, are continuous and $\mathscr{J}_i(\mathfrak{P},2\alpha) = 2\mathscr{J}_i(\mathfrak{P},\alpha), (i = 1,\cdots,n)$, for each $\alpha \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$, then the mappings $\mathscr{J}_i : \mathscr{X}' \times \mathscr{X} \to \mathscr{X}, (i = 1,\cdots,n)$, are ternary antiderivations.

Proof. The proof follows from Theorem 2.5 by letting $(\mathfrak{T}_1,\cdots,\mathfrak{T}_n) = (\frac{32}{33},\cdots,\frac{32}{33})$ and

$$\begin{aligned}
&\operatorname{diag}\Bigg[\ \underbrace{\varphi_{j_1}}_{1\leqslant j_1\leqslant n}(\alpha,\beta,\gamma),\ldots,\ \underbrace{\varphi_{j_n}}_{1\leqslant j_n\leqslant n}(\alpha,\beta,\gamma)\ \Bigg] \\
&=: \operatorname{diag}\Bigg[\ \underbrace{\varphi^{Ⓢ}_{j_1}}_{1\leqslant j_1\leqslant n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big),\ldots,\ \underbrace{\varphi^{Ⓢ}_{j_n}}_{1\leqslant j_n\leqslant n}\Big(\|[[\alpha,\alpha,\beta],\beta,\gamma]\|\Big)\ \Bigg]
\end{aligned}$$

for each $\alpha,\beta,\gamma \in \mathscr{X}, \mathfrak{P} \in \mathscr{X}'$. □

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